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Grade 7 & 8 Math Circles

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Algebra

Introduction

When evaluating mathematical expressions it is important to remember that there is only one right answer. Because of this, there is a strict set of rules that must be followed by all mathematicians so that we can all agree and understand why an answer is correct.

Order of Operations

When working on more complicated mathematical expressions that involve multiple operations and many steps it is very important to do the math *in the right order* to ensure that you get the right answer.

What is the right order?

1. Begin by simplifying everything inside of **brackets**.
2. Evaluate anything with **exponents**.
3. Starting from the left, work out all of the **multiplication** and **division**, *whichever comes first.*
4. Starting from the left, finish by evaluating all of the **addition** and **subtraction**, *whichever comes first.*

This can be summarized with the acronym **BEDMAS**:

Brackets **E**xponents **D**ivision **M**ultiplication **A**ddition **S**ubtraction

Remember that division and multiplication are evaluated in the same step, even though D appears before M in the acronym; *division is no more important than multiplication.* Similarly, *addition is no more important than subtraction.*

Example

$$\begin{aligned} 2^2 \times 6 - 35 \div (5+2) &= 2^2 \times 6 - 35 \div 7 \\ &= 4 \times 6 - 35 \div 7 \\ &= 24 - 5 \\ &= 19 \end{aligned}$$

Exercises I

Evaluate the following expressions.

(a) $15 \div 3 - 1$

(b) $7 \times 3 + 4$

(c) $48 \div 4 + 4 \times 5$

(d) $48 \div (4+4) \times 5$

(e) $11 - 6^2 \div 12 + (4 + 5 \times 2) \div 7$

Distributive Property

The Distributive Property is used when multiplying a sum by a single term or another sum.

$$a \times (b+c) \qquad\qquad (a+b) \times (c+d)$$

To evaluate $a \times (b+c)$, multiply every term in the sum by the constant in front. That is,

$$a \times (b+c) = a \times b + a \times c.$$

Examples

$$\begin{aligned} 3 \times (2+5) &= 3 \times 2 + 3 \times 5 \\ &= 6 + 15 \\ &= 21 \end{aligned}$$

We can check that this is right by using our BEDMAS rules:

$$\begin{aligned} 3 \times (2+5) &= 3 \times 7 \\ &= 21 \end{aligned}$$

This might seem like more work, but it can be really useful when doing mental math:

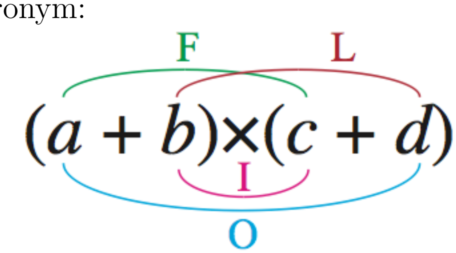
$$\begin{aligned} 8 \times 43 &= 8 \times (40+3) \\ &= 8 \times 40 + 8 \times 3 \\ &= 320 + 24 \\ &= 344 \end{aligned}$$

We can use this idea to also evaluate $(a+b)\times(c+d)$:

$$\begin{aligned}(a+b)\times(c+d) &= a\times(c+d)+b\times(c+d)\\ &= a\times c+a\times d+b\times c+b\times d\\ &= ac+ad+bc+bd\end{aligned}$$

As you can see, every term in the first sum is multiplied by every term in the second sum. To remember this, the mnemonic **FOIL** is used. FOIL is an acronym:

Firsts	Multiply the first term in each sum (a and c).
Outsides	Multiply the two terms on the outsides (a and d).
Insides	Multiply the innermost terms (b and c).
Lasts	Multiply the last term in each sum (b and d).



FOIL can also be represented visually in a chart:

	a	b
c	ac	bc
d	ad	bd

Reminder: When working with the Distributive Property, it is important to remember how to multiply and divide negative and positive numbers.

- a **positive** multiplied or divided by a **positive** will result in a **positive** number
- a **negative** multiplied or divided by a **negative** will result in a **positive** number
- a **positive** multiplied or divided by a **negative** will result in a **negative** number

Example

$$\begin{aligned}(2+5)\times(6-3) &= (2)(6)+(2)(-3)+(5)(6)+(5)(-3)\\ &= 12-6+30-15\\ &= 21\end{aligned}$$

	2	5
6	12	30
-3	-6	-15

$12+30-6-15=21$

We can check that this is right by using our BEDMAS rules:

$$\begin{aligned}(2+5)\times(6-3) &= 7\times 3\\ &= 21\end{aligned}$$

Exercises II

Evaluate the following expressions without a calculator.

(a) 8×27

(b) 14×7

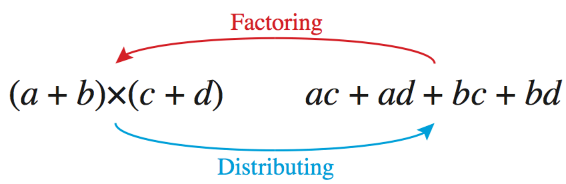
(c) 14×27

(d) 89×76

(e) $(a + b + c) \times (d + e + f)$

Factorization

Factoring is the opposite of distributing. That is, factoring breaks down an expression into the product of its simplest components.



Examples

1. The number 21 can be written as $21 = 3 \times 7$. We say: "3 and 7 are factors of 21". Because 3 and 7 are both prime numbers we call this a *prime factorization*.

2. The number 48 can be written as $48 = 4 \times 12$. Both 4 and 12 are factors of 48, but neither are prime numbers, so we must find the prime factorizations of those numbers first.

$$4 = 2 \times 2 \qquad \begin{aligned} 12 &= 2 \times 6 \\ &= 2 \times 2 \times 3 \end{aligned}$$

   Notice that 2 and 3 are both prime numbers. Thus the prime factorization of 48 is $48 = 2 \times 2 \times 2 \times 2 \times 3$ or $48 = 2^4 \times 3$

3. You can factor a common divisor out of a sum: $ab + ac = a(b + c)$.
   Consider the sum $6 + 10$.
   6 and 10 are both even numbers and therefore divisible by 2. Thus:

$$\begin{aligned} 6 + 10 &= 2 \times 3 + 2 \times 5 \\ &= 2(3 + 5) \end{aligned}$$

Exercises III

Find the prime factorizations of the following integers.

(a) 35

(b) 36

(c) 144

Factor a common divisor out of the following sums.

(d) $14 + 63 + 35$

(e) $6 + 54 + 12 + 48 + 18 + 42 + 24 + 36 + 30$

Solving Equations

In ten years, Matt will be twice as old as he was six years ago. How old is he right now?

How can you solve this problem?

You *could* "guess and check", where you start guessing numbers and check to see if they satisfy the conditions. But this is inefficient; it takes too long and doesn't work with more complicated problems.

Instead, you can set up an equation with a variable representing Matt's age right now. A **variable** is a symbol that represents an unknown quantity. In algebra it is often our goal to isolate a variable so that it is no longer unknown.

STEPS FOR SOLVING:

1. Determine what you are trying to isolate/solve for.

2. Simplify the equation as much as possible by adding and subtracting like terms.

   Like terms are terms in a mathematical equation that have the exact same variables; only their coefficients are different.

   You can think of it like adding apples and oranges. If I have 3 apples plus 2 apples plus 5 oranges plus 1 orange, I actually have 5 apples and 6 oranges. Another, more mathematical, example:

   $$5 + x + 3y - 2 - y + 2x = 3 + 3x + 2y$$

3. Isolate the desired variable on one side of the equal sign and everything else on the other side by performing opposite operations *in reverse BEDMAS order*. The goal of isolating a variable, say x, is to obtain the form $x = ...$ or $... = x$. Notice that x is positive with a coefficient of 1. There should be no other xs on the other side of the equal sign.

Original Operation			Opposite Operation
Addition	$+$	$-$	Subtraction
Subtraction	$-$	$+$	Addition
Division	$\div$	$\times$	Multiplication
Multiplication	$\times$	$\div$	Division

Reverse BEDMAS (**SAMDEB**):

Subtraction **A**ddition **M**ultiplication **D**ivision **E**xponents **B**rackets

REMINDER: Just as before, addition and subtraction have the same priority; as do multiplication and division.

> When performing opposite operations, what you do to one side of the equation you **must** do to the other side of the equation.

Examples

Let's solve for x.

1.
$$\begin{aligned} x - 5 &= 7 \\ x - 5 + 5 &= 7 + 5 \\ x &= 12 \end{aligned}$$

2.
$$\begin{aligned} x + 3 &= 10 \\ x + 3 - 3 &= 10 - 3 \\ x &= 7 \end{aligned}$$

3.
$$\begin{aligned} 15 - x &= 8 \\ 15 - x + x &= 8 + x \\ 15 - 8 &= 8 + x - 8 \\ 7 &= x \end{aligned}$$

4.
$$\begin{aligned} 2x &= 10 \\ \frac{2x}{2} &= \frac{10}{2} \\ x &= 5 \end{aligned}$$

5.
$$\begin{aligned} \frac{x}{3} &= 2 \\ \frac{x}{3} \times 3 &= 2 \times 3 \\ x &= 6 \end{aligned}$$

6.
$$\begin{aligned} 4x - 1 &= 7 \\ 4x - 1 + 1 &= 7 + 1 \\ \frac{4x}{4} &= \frac{8}{4} \\ x &= 2 \end{aligned}$$

7.
$$\begin{aligned} 8 - 2x &= 2 \\ 8 - 2x + 2x &= 2 + 2x \\ 8 - 2 &= 2 + 2x - 2 \\ \frac{6}{2} &= \frac{2x}{2} \\ 3 &= x \end{aligned}$$

8.
$$\begin{aligned} x + 10 &= 2(x - 6) \\ x + 10 + 12 &= 2x - 12 + 12 \\ x + 22 - x &= 2x - x \\ 22 &= x \end{aligned}$$

Matt is 22.

Exercises IV

Solve for x. Show all of your steps.

(a) $x + 3 = 2$

(b) $2 - x = -3$

(c) $-3x + 7 = -8$

(d) $\frac{x}{10} + 5 = 7$

(e) $2 - x + 8 - y = -x + y + 3 - 2 - x$

(f) $\frac{10}{x} + 1 = \frac{9}{4}$

Solving Systems of Equations

Amanda has two fair options at the toy shop.

(1) She can buy one deck of Pokémon cards and one Rubik's Cube for \$20.

(2) Or for \$7, she can buy two decks of Pokémon cards but she also has to trade in the Rubik's Cube she has at home.

What are the prices of a deck of Pokémon cards and a Rubik's Cube?

How can you solve this problem?

Notice that we have two unknowns: the price of a deck of Pokémon cards (x) and the price of a Rubik's Cube (y). We also have two different options that can be represented by two different equations:

(1) one deck + one cube = \$20. Mathematically, $1x + 1y = 20$.

(2) two decks = \$7 + one cube. Mathematically, $2x = 7 + 1y$ or $2x - 1y = 7$.

The goal in this situation is to find x and y such that both equations are satisfied.

A System of Equations is a set of multiple equations dealing with multiple variables. You have to work with all of the equations at the same time in order to solve for the variables.

Example

A system of equations looks like:

$$x + y = 20 \tag{1}$$
$$2x - y = 7 \tag{2}$$

We will use everything we have covered today, plus **substitution**, to solve this system.We must find x and y such that both equations are satisfied.

Steps for Solving:

1. Pick an equation and solve for a variable of your choice in terms of the other variable.
2. Replace the variable in the second equation with the expression found above.
3. Solve for the remaining variable in the equation and substitute that expression into the initial equation you chose and solve for the last variable.

This will become more clear in the following example.

Example

$$x + y = 20 \tag{1}$$
$$2x - y = 7 \tag{2}$$

First, solve (1) for y.

$$\begin{aligned} x + y - x &= 20 - x \\ y &= 20 - x \end{aligned}$$

Now replace y in (2) with $20 - x$.

$$2x - (20 - x) = 7$$

Solve this for x.

$$\begin{aligned} 2x - (20 - x) &= 7 \\ 2x - 20 + x &= 7 \\ 3x - 20 + 20 &= 7 + 20 \\ \frac{3x}{3} &= \frac{27}{3} \\ x &= 9 \end{aligned}$$

Substitute $x = 9$ into (1) and solve for y.

$$\begin{aligned} (9) + y &= 20 \\ 9 + y - 9 &= 20 - 9 \\ y &= 11 \end{aligned}$$

So we found $x = 9$ and $y = 11$. Substitute these values into equations (1) and (2) to make sure they are both satisfied.

$$\begin{aligned} x + y &= 9 + 11 \\ &= 20 \end{aligned} \qquad (1)$$

$$\begin{aligned} 2x - y &= 2(9) - 11 \\ &= 18 - 11 \\ &= 7 \end{aligned} \qquad (2)$$

Therefore $x = 9$ and $y = 11$ are the solutions to the system.

Exercises V

Solve the systems. Show all of your steps.

(a) $x - 2y = 6$
$3x + y = 25$

(b) $2x + 3y = 17$
$-x - y = -4$

Problem Set

1. Evaluate the following expressions.

   (a) $(2+3)\times 5^2-(15-20\div 5)^2$

   (b) $((-2+2^3)\times 4-(10\div 2)^2)^2$

   (c) $(2^3\times 3^2-9^2)\div 3$

   (d) $2(30-(10-(4+12\div 4))^3)-6$

2. How much smaller is the sum of 3 and 7 than the product of 4 and 8?

3. If $a=b=c=d$ and $a+b+c+d=16$, what is the value of $a\times b\times c\times d$?

4. *You are a contestant on the popular European game show COUNTDOWN. During a "Numbers Round" you draw six numbers,

   $\boxed{75}\ \boxed{2}\ \boxed{5}\ \boxed{6}\ \boxed{1}\ \boxed{4}$

   and a target.

   $\boxed{273}$

   Your goal is to use your numbers and four arithmetic operations (addition, subtraction, multiplication, and division) in order to reach your target. You can only use each number once, but you do not have to use all of your numbers. How close can you get?

5. If $\odot$ is an operation defined as $p\odot q=p^2+3pq-2q+1$, what is the value of $7\odot 5$?

6. Evaluate the following products without a calculator.

   (a) 3×57

   (b) 5×371

   (c) $\frac{1}{2}\times 1256$

   (d) 23×32

   (e) 17×142

7. Solve for x. Show all of your work.

   (a) $2x+3=-7$

   (b) $x+4-2x=3x+20$

   (c) $\frac{x}{4}+3=\frac{15}{4}$

8. The product of 2, 4, 6, and x is equal to its sum. What is value of x?

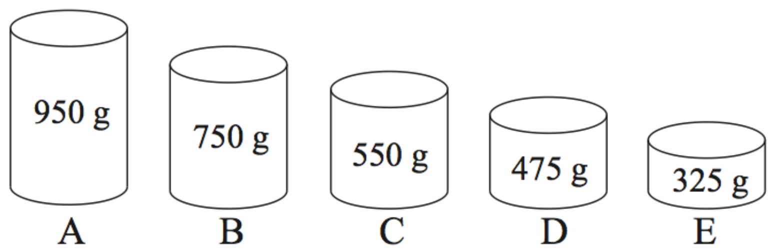
9. Clark scored a total of 36 points in his basketball team's first four games. He scored $\frac{1}{4}$ of these points in the first game, $\frac{1}{6}$ of these points in the second game, and $\frac{2}{9}$ of the points in the third game. How many points did he score in the fourth game?

10. If $x-2y=-12$ and $\frac{x}{2}+y=8$, then what is the value of $\frac{y}{3x+8}$?

11. Expand and simplify the expressions.

   (a) $(a + b)^2$ (Note: $(a + b)^2 = (a + b) \times (a + b)$)
   (b) $(a - b)^2$
   (c) $(a + b) \times (b - a)$
   (d) $(a + b) \times (a - b)$

12. *Each of the five tins below contains either coffee, cocoa, or powdered milk. There is twice as much coffee as cocoa by total weight. No three tins contain the same item. Which tin is the only one containing cocoa?



Answers

I	II	III	IV	V
(a) 4	(a) 216	(a) 5×7	(a) $x = -1$	(a) $x = 8$, $y = 1$
(b) 25	(b) 98	(b) $3 \times 3 \times 2 \times 2$	(b) $x = 5$	(b) $x = -5$, $y = 9$
(c) 32	(c) 378	(c) $3 \times 2 \times 2 \times 3 \times 2 \times 2$	(c) $x = 5$	
(d) 30	(d) 6764	(d) $7(2 + 9 + 5)$	(d) $x = 20$	
(e) 10	(e) $ad+ae+af+bd+be+bf+cd+ce+cf$	(e) $6(1+9+2+8+3+7+4+6+5)$	(e) $x = 2y - 9$	
			(f) $x = 8$	

Problem Set

1. (a) 4 (b) 1 (c) -3 (d) 0
2. 22
3. 256
4. Answers may vary. See solutions.
5. 145
6. (a) 171 (b) 1855 (c) 628 (d) 736 (e) 2414
7. (a) $x = -5$ (b) $x = -4$ (c) $x = 3$
8. $x = \dfrac{12}{47}$
9. 13 points
10. $\dfrac{1}{2}$
11. (a) $a^2 + 2ab + b^2$ (b) $a^2 - 2ab + b^2$ (c) $b^2 - a^2$ (d) $a^2 - b^2$
12. B